

Hermiteov identitet i primjene

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Poznati francuski matematičar Charles Hermite (1822.–1901.) je davne 1884. dokazao identitet iz teorije brojeva (vidi [4]), vezan za funkciju najveće cijelo, koji ima primjene, među ostalim, u brojnim zadacima koji se pojavljuju na matematičkim natjecanjima.

Neka je $x \in \mathbf{R}$. Tada s $\lfloor x \rfloor$ označavamo najveći cijeli broj koji nije veći od x , a s $\{x\} = x - \lfloor x \rfloor$ razlomljeni dio od x , $0 \leq \{x\} < 1$.

Teorem 1 (Hermite). *Za svaki realan broj x i prirodan broj n vrijedi:*

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor.$$

Prvi dokaz. Ako je $\{x\} = 0$, onda je $x \in \mathbf{Z}$ i tvrdnja očigledno vrijedi jer $\left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor x \rfloor$, $k = 0, \dots, n-1$.

Pretpostavimo $0 < \{x\} < 1$. Tada zbog

$$\lfloor x \rfloor \leq \left\lfloor x + \frac{1}{n} \right\rfloor \leq \cdots \leq \left\lfloor x + \frac{n-1}{n} \right\rfloor \leq \lfloor x \rfloor + 1$$

postoji i , $1 \leq i \leq n-1$ takav da je

$$\lfloor x \rfloor = \left\lfloor x + \frac{1}{n} \right\rfloor = \cdots = \left\lfloor x + \frac{i-1}{n} \right\rfloor$$

i

$$\left\lfloor x + \frac{i}{n} \right\rfloor = \left\lfloor x + \frac{i+1}{n} \right\rfloor = \cdots = \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor x \rfloor + 1.$$

Sada je

$$\begin{aligned} \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor &= i\lfloor x \rfloor + (n-i)(\lfloor x \rfloor + 1) \\ &= n\lfloor x \rfloor + n - i. \end{aligned} \quad (1)$$

Dalje, i zadovoljava

$$\{x\} + \frac{i-1}{n} < 1 \quad \text{ i } \quad \{x\} + \frac{i}{n} \geq 1,$$

sve zajedno

$$\frac{n-i}{n} \leq \{x\} < \frac{n-i+1}{n}.$$

Donja i gornja granica za nx sada glasi

$$n\lfloor x \rfloor + n - i \leq n\lfloor x \rfloor + n\{x\} = nx < n\lfloor x \rfloor + n - i + 1,$$

i iz (1) imamo tvrdnju. \square

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Japanski matematičar Matsuoka, u [5], je dao elegantan dokaz Hermiteovog identiteta:

Drugi dokaz. Neka je $f(x)$ razlika između lijeve i desne strane našeg identiteta. Tada vrijedi:

$$\begin{aligned} f\left(x + \frac{1}{n}\right) &= \left\lfloor x + \frac{1}{n} \right\rfloor + \dots + \left\lfloor x + \frac{1}{n} + \frac{n-1}{n} \right\rfloor - \left\lfloor n \cdot \left(x + \frac{1}{n}\right) \right\rfloor \\ &= \left\lfloor x + \frac{1}{n} \right\rfloor + \dots + \left\lfloor x + \frac{n-1}{n} \right\rfloor + \lfloor x + 1 \rfloor - \lfloor nx + 1 \rfloor = f(x), \quad \forall x \in \mathbf{R}. \end{aligned}$$

Dakle, funkcija f je periodička s periodom $\frac{1}{n}$. Prema tome, dovoljno ju je promatrati na intervalu $\left[0, \frac{1}{n}\right)$. No, za sve te vrijednosti ona je jednaka nuli, pa je $f(x) = 0$, $\forall x \in \mathbf{R}$. \square

Pokažimo sada primjenu Hermiteovog identiteta na nekim zadacima. Sljedeći zadatak je zadao poznati indijski matematičar Ramanujan (vidi [6]), a pojavio se i kao problem u MFL-u broju 205/1 iz 2001. godine.

Zadatak 1. Dokažite da za svaki $n \in \mathbf{N}$ vrijedi

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor.$$

Prvo rješenje.

$$\begin{aligned} \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor &= \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n}{6} \right\rfloor + \left(\left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n}{6} + \frac{1}{3} \right\rfloor + \left\lfloor \frac{n}{6} + \frac{2}{3} \right\rfloor \right) \\ (\text{Hermite}) &= \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor 2 \cdot \frac{n}{6} \right\rfloor - \left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\ (\text{Hermite}) &= \left\lfloor \frac{n}{6} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor. \quad \square \end{aligned}$$

Drugo rješenje. Za drugo rješenje zadatka koristimo gornju Matsuokinu ideju: funkcija $f: \mathbf{N} \rightarrow \mathbf{Z}$

$$f(n) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n+3}{6} \right\rfloor$$

je perioda 6 jer $f(n) = f(n+6)$, pa je dovoljno provjeriti da je $f(n) = 0$, za $n = 0, 1, 2, 3, 4, 5$. \square

Ponekad trik s periodičnosti nije očit, što pokazuje sljedeći primjer. U tom slučaju je zgodnije koristiti Hermiteov identet.

Zadatak 2. Dokažite da za svaki $n \in \mathbf{Z}$ vrijedi

$$\left| \left\lfloor -\frac{n}{2} \right\rfloor \right| = \left| \left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor \right|.$$

Rješenje.

$$\left| \left\lfloor -\frac{n}{2} \right\rfloor \right| = (\text{Hermite}) = \left| -n - \left\lfloor -\frac{n}{2} + \frac{1}{2} \right\rfloor \right| = \left| n + \left\lfloor -\frac{n}{2} + \frac{1}{2} \right\rfloor \right| = \left| \left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor \right|. \quad \square$$

Zadatak 3. Dokažite da za sve realne brojeve x vrijedi

$$\left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor + \left\lfloor 3x + \frac{1}{3} \right\rfloor + \left\lfloor 3x + \frac{2}{3} \right\rfloor + \dots + \left\lfloor 3^n x + \frac{1}{3} \right\rfloor + \left\lfloor 3^n x + \frac{2}{3} \right\rfloor = \lfloor 3^{n+1} x \rfloor - \lfloor x \rfloor.$$

Rješenje. Uzastopnom primjenom Hermiteova identiteta za $n = 3$ imamo:

$$\left. \begin{aligned} \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor &= \lfloor 3x \rfloor - \lfloor x \rfloor \\ \left\lfloor 3x + \frac{1}{3} \right\rfloor + \left\lfloor 3x + \frac{2}{3} \right\rfloor &= \lfloor 3^2 x \rfloor - \lfloor 3x \rfloor \\ &\vdots \\ \left\lfloor 3^n x + \frac{1}{3} \right\rfloor + \left\lfloor 3^n x + \frac{2}{3} \right\rfloor &= \lfloor 3^{n+1} x \rfloor - \lfloor 3^n x \rfloor \end{aligned} \right\}$$

Zbrajanjem ovih jednakosti slijedi tvrdnja zadatka. \square

Zadatak 4. Izračunajte sumu

$$\sum_{0 \leq i < j \leq n} \left\lfloor \frac{x+i}{j} \right\rfloor.$$

Rješenje.

$$\sum_{0 \leq i < j \leq n} \left\lfloor \frac{x+i}{j} \right\rfloor = \sum_{j=1}^n \left(\sum_{i=1}^{j-1} \left\lfloor \frac{x+i}{j} \right\rfloor \right) = (\text{Hermite}) = \sum_{j=1}^n \left\lfloor j \cdot \frac{x}{j} \right\rfloor = n \lfloor x \rfloor \quad \square$$

Zadatak 5. (10. MMO) Izračunajte:

$$\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n+4}{8} \right\rfloor + \dots + \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor + \dots$$

gdje je n nenegativan cijeli broj.

Rješenje.

$$\begin{aligned} S_k &= \left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{4} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{n}{8} + \frac{1}{2} \right\rfloor + \dots + \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor = (\text{Hermite}, n=2) \\ &= \left(\lfloor n \rfloor - \left\lfloor \frac{n}{2} \right\rfloor \right) + \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor \right) + \left(\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor \right) + \dots + \left(\left\lfloor \frac{n}{2^{k-1}} \right\rfloor - \left\lfloor \frac{n}{2^k} \right\rfloor \right) \\ &= n - \left\lfloor \frac{n}{2^k} \right\rfloor. \end{aligned}$$

Sada je

$$S = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \left(n - \left\lfloor \frac{n}{2^k} \right\rfloor \right) = n. \quad \square$$

Zadatak 6. (Srbija i Crna Gora 2004.) Niz $\{a_n\}$ zadan je uvjetima $a_1 = x \in \mathbf{R}$ i $3a_{n+1} = a_n + 1$, za $n \geq 1$. Neka je:

$$A = \sum_{n=1}^{\infty} \left\lfloor a_n - \frac{1}{6} \right\rfloor, \quad B = \sum_{n=1}^{\infty} \left\lfloor a_n + \frac{1}{6} \right\rfloor.$$

Izračunajte zbroj $A + B$ u ovisnosti o x .

Rješenje. Označimo k -tu parcijalnu sumu

$$S_k = \sum_{n=1}^k \left(\left\lfloor a_n - \frac{1}{6} \right\rfloor + \left\lfloor a_n + \frac{1}{6} \right\rfloor \right).$$

Iz rekurzije $3a_{n+1} = a_n + 1$ imamo

$$\begin{aligned} 3a_2 &= a_1 + 1 \\ 9a_3 &= 3a_2 + 3 \\ &\vdots \\ 3^{n-2}a_{n-1} &= 3^{n-3}a_{n-2} + 3^{n-3} \\ 3^{n-1}a_n &= 3^{n-2}a_{n-1} + 3^{n-2}. \end{aligned}$$

Teleskopiranjem gornjih jednakosti:

$$a_n = \frac{1}{3^{n-1}} \left(x - \frac{1}{2} \right) + \frac{1}{2}.$$

Računamo:

$$\left\lfloor a_n - \frac{1}{6} \right\rfloor = \left\lfloor \frac{1}{3^{n-1}} \left(x - \frac{1}{2} \right) + \frac{1}{3} \right\rfloor, \quad \left\lfloor a_n + \frac{1}{6} \right\rfloor = \left\lfloor \frac{1}{3^{n-1}} \left(x - \frac{1}{2} \right) + \frac{2}{3} \right\rfloor.$$

Sada je, korištenjem Hermiteovog identiteta,

$$\begin{aligned} S_k &= \sum_{n=1}^k \left(\left\lfloor \frac{1}{3^{n-2}} \left(x - \frac{1}{2} \right) \right\rfloor - \left\lfloor \frac{1}{3^{n-1}} \left(x - \frac{1}{2} \right) \right\rfloor \right) \\ &= \left\lfloor 3 \left(x - \frac{1}{2} \right) \right\rfloor - \left\lfloor \frac{1}{3^{k-1}} \left(x - \frac{1}{2} \right) \right\rfloor. \end{aligned}$$

Ako je $x \geq \frac{1}{2}$, za dovoljno velike k je $\left\lfloor \frac{1}{3^{k-1}} \left(x - \frac{1}{2} \right) \right\rfloor = 0$, a ako je $x < \frac{1}{2}$, za dovoljno velike k je $\left\lfloor \frac{1}{3^{k-1}} \left(x - \frac{1}{2} \right) \right\rfloor = -1$. Sada je

$$A + B = \left\lfloor 3 \left(x - \frac{1}{2} \right) \right\rfloor - \lim_{k \rightarrow \infty} \left\lfloor \frac{1}{3^{k-1}} \left(x - \frac{1}{2} \right) \right\rfloor = \begin{cases} \left\lfloor 3 \left(x - \frac{1}{2} \right) \right\rfloor, & x \geq \frac{1}{2} \\ \left\lfloor 3 \left(x - \frac{1}{2} \right) \right\rfloor + 1, & x < \frac{1}{2}. \end{cases}$$

□

Zadatak 7. Dokaži da za svako $n \in \mathbf{N}$ vrijedi

$$\left\lfloor \frac{3n+3}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} \right\rfloor + \left\lfloor \frac{n+5}{13} \right\rfloor + \left\lfloor \frac{n+9}{13} \right\rfloor. \quad (2)$$

Rješenje. Prema Hermiteovom identitetu vrijedi

$$\left\lfloor \frac{3n+3}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} \right\rfloor + \left\lfloor \frac{n+1}{13} + \frac{1}{3} \right\rfloor + \left\lfloor \frac{n+1}{13} + \frac{2}{3} \right\rfloor.$$

Pokazat ćemo da vrijedi

$$\left\lfloor \frac{n+5}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} + \frac{1}{3} \right\rfloor, \quad \left\lfloor \frac{n+9}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} + \frac{2}{3} \right\rfloor.$$

Prvo $\frac{n+5}{13} = \frac{n+1}{13} + \frac{1}{3} - \frac{1}{39}$, pa je

$$\left\lfloor \frac{n+5}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} + \frac{1}{3} \right\rfloor \iff \frac{1}{39} \leq \left\{ \frac{n+1}{13} + \frac{1}{3} \right\} < 1. \quad (3)$$

Kako za svaki $n \in \mathbf{N}$ vrijedi $n = 13k + r$, $r = 0, 1, \dots, 12$, sada je dovoljno provjeriti (3) za $n = 0, 1, \dots, 12$.

Slično, $\frac{n+5}{13} = \frac{n+1}{13} + \frac{2}{3} - \frac{2}{39} \implies$

$$\left\lfloor \frac{n+9}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} + \frac{2}{3} \right\rfloor \iff \frac{2}{39} \leq \left\{ \frac{n+1}{13} + \frac{2}{3} \right\} < 1. \quad (4)$$

Kako (4) vrijedi za $n = 0, 1, \dots, 12$, zaključujemo da (2) vrijedi za sve $n \in \mathbf{N}$. \square

Napomena. Zadatak se može riješiti i na drugi način, koristeći gore spomenutu tehniku:

Definirajmo funkciju

$$f(n) = \left\lfloor \frac{3n+3}{13} \right\rfloor - \left\lfloor \frac{n+1}{13} \right\rfloor - \left\lfloor \frac{n+5}{13} \right\rfloor - \left\lfloor \frac{n+9}{13} \right\rfloor.$$

Kako je $f(n) = f(n+13)$, dovoljno je samo provjeriti $f(n) = 0$, $n = 0, 1, \dots, 12$. \square

Jedno zanimljivo pitanje: može li se u (2) izraz $\frac{3n+3}{13}$ zamijeniti s $\frac{3n+2}{13}$, tj. vrijedi li

$$\left\lfloor \frac{3n+2}{13} \right\rfloor = \left\lfloor \frac{n+1}{13} \right\rfloor + \left\lfloor \frac{n+5}{13} \right\rfloor + \left\lfloor \frac{n+9}{13} \right\rfloor?$$

Da bi odgovorili na ovo pitanje koristimo Hermiteov identitet:

$$\left\lfloor \frac{3n+2}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} \right\rfloor + \left\lfloor \frac{n+\frac{2}{3}}{13} + \frac{1}{3} \right\rfloor + \left\lfloor \frac{n+\frac{2}{3}}{13} + \frac{2}{3} \right\rfloor$$

i slijedeći gornju metodu, treba provjeriti,

$$\left\lfloor \frac{n+1}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} \right\rfloor, \quad \left\lfloor \frac{n+5}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} + \frac{1}{3} \right\rfloor, \quad \left\lfloor \frac{n+9}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} + \frac{2}{3} \right\rfloor.$$

Međutim, zbog $\frac{n+1}{13} = \frac{n+\frac{2}{3}}{13} + \frac{1}{39}$ zaključujemo

$$\left\lfloor \frac{n+1}{13} \right\rfloor = \left\lfloor \frac{n+\frac{2}{3}}{13} \right\rfloor \iff \left\{ \frac{n+\frac{2}{3}}{13} \right\} + \frac{1}{39} < 1$$

a to ne vrijedi za $n = 12, 24, 36, \dots$

O generiranju zadatka 6 uočimo da vrijedi $\frac{4}{13} \approx \frac{1}{3}$, $\frac{8}{13} \approx \frac{2}{3}$, a to su razlomci koji se javljaju u Hermiteovom identitetu za $n = 3$.

U zaključku rada ističemo da korištenjem Hermiteovog identiteta imamo bolji uvid u problem, njegovu analizu i metodu generiranja sličnih zadataka.

Zadaci za vježbu

1. Dokažite da za svako $x \in \mathbf{R}$ i $n \in \mathbf{N}$ vrijedi

$$\{x\} + \left\{x + \frac{1}{n}\right\} + \left\{x + \frac{2}{n}\right\} + \dots + \left\{x + \frac{n-1}{n}\right\} = \{nx\} + \frac{n-1}{2}.$$

2. Dokažite da za svaki realan broj x vrijedi

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor + \left\lfloor 2x + \frac{1}{2} \right\rfloor + \dots + \left\lfloor 2^{2015}x + \frac{1}{2} \right\rfloor = \lfloor 2^{2016}x \rfloor.$$

3. Neka je r realan broj za koji vrijedi

$$\left\lfloor r + \frac{19}{100} \right\rfloor + \left\lfloor r + \frac{20}{100} \right\rfloor + \dots + \left\lfloor r + \frac{91}{100} \right\rfloor = 546.$$

Nadite $\lfloor 100r \rfloor$. (Rezultat: $\lfloor 100r \rfloor = 743$.)

4. Dokažite da za svaki $n \in \mathbf{N}$ vrijedi

$$\left\lfloor \frac{n-2^0}{2} \right\rfloor + \left\lfloor \frac{n-2^1}{2^2} \right\rfloor + \left\lfloor \frac{n-2^2}{2^3} \right\rfloor + \dots + \left\lfloor \frac{n-2^{n-1}}{2^n} \right\rfloor = 0.$$

5. Dokažite da se u jednakosti

$$n = \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \dots + \frac{n}{2^k} + \dots$$

svi razlomci mogu zamijeniti s njima najbližim cijelim brojem:

$$n = \left\langle \frac{n}{2} \right\rangle + \left\langle \frac{n}{4} \right\rangle + \left\langle \frac{n}{8} \right\rangle + \dots + \left\langle \frac{n}{2^k} \right\rangle + \dots$$

gdje je $\langle r \rangle = \left\lfloor r + \frac{1}{2} \right\rfloor$.

6. Izračunajte sumu

$$\sum_{k=1}^{\infty} \left\lfloor \frac{x+3^k}{3^{k+1}} + \left\lfloor \frac{x+2 \cdot 3^k}{3^{k+1}} \right\rfloor \right\rfloor.$$

(Rezultat: $\left\lfloor \frac{x}{3} \right\rfloor$.)

7. Dokažite da za svaki realan broj x i prirodan broj m vrijedi:

$$\sum_{n=1}^m \left(\left\{ 2^n x + \frac{1}{2} \right\} - \frac{1}{2} \right) \leq 1.$$

8. Dokažite da za svako $n \in \mathbb{N}$ vrijedi

$$\left\lfloor \frac{3n+8}{25} \right\rfloor = \left\lfloor \frac{n+2}{25} \right\rfloor + \left\lfloor \frac{n+11}{25} \right\rfloor + \left\lfloor \frac{n+19}{25} \right\rfloor.$$

9. Ako je:

$$A = \sum_{k=0}^{2000} \left\lfloor \frac{3^k + 2000}{3^{k+1}} \right\rfloor \quad \text{i} \quad B = \sum_{k=0}^{2000} \left\lfloor \frac{3^k - 2000}{3^{k+1}} \right\rfloor,$$

izračunajte $A - B$. (Rezultat: $A - B = 2000$.)

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